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EIGENFUNCTION LOCALIZATION IN THE QUANTIZED RIGID BODY

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1. Introduction

It has long been known [5],[10],[12],[16],[17] that given a stable, closed, elliptic geodesic γ, one can associate with this curve a sequence of quasimodes ϕ_n for the corresponding Laplace operator $-\Delta$, in the sense that the ϕ_n have microsupport in a tube of width $\mathcal{O}(n^{-1/2})$ about γ and decay exponentially outside this tube. On the other hand, in the unstable, hyperbolic case, it is known [13] that under suitable hypotheses, one can associate complex resonances with hyperbolic orbits. However, analogous general results are not known for eigenvalues and eigenfunctions (see, however [4],[6],[7]. In this paper we focus on a specific paradigm; namely, that of the asymmetric rigid body reduced at an S^1 Noether symmetry. The corresponding quantum system on S^2 is integrable with the classical Lamé harmonics as joint eigenfunctions [20]. The classical system inherits a natural hyperbolic geodesic Γ corresponding to the unstable rotation about the middle-length inertial axis. Given the quantum Hamiltonian $\mathcal{H}$, we show that there is a sequence of L^2-normalized eigenfunctions, ψ_n, with L^∞ norm concentrated along Γ. More precisely, let $\Gamma(n^{-1})$ denote a tube of width $\mathcal{O}(n^{-1})$ about Γ, and let $V_j; j = 1,2,3,4$ denote arbitrarily small (but fixed) disconnected neighbourhoods about the four umbilic points on Γ. Our main results are:

$$\|\psi_n\|_{L^\infty(\Gamma(n^{-1})-\bigcup_j V_j)} = C\frac{n^{1/4}}{\log n} + \mathcal{O}\left(\frac{n^{1/4}}{(\log n)^2}\right),$$

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$$\|\psi_n\|_{L^\infty(n^{-1}V_j)} = C\frac{n^{1/2}}{\log n} + \mathcal{O}\left(\frac{n^{1/2}}{(\log n)^2}\right),$$

and $\|\psi_n\|_{L^\infty} = \mathcal{O}(1/\log n)$ outside an arbitrarily small (but fixed) neighbourhood of Γ. Thus we encounter eigenfunction accumulation along the hyperbolic geodesic Γ, with additional intensity near the umbilic points. This corresponds to a focusing effect for geodesic flow at the classical level (see Proposition 2). Our analysis is based on a fundamental construction of Helffer and Sjöstrand [14] (see Theorem 1) and subsequent work of März [15] on the behaviour of the Floquet spectrum of a one-dimensional Schrödinger operator (with periodic, real-analytic potential) near the potential maximum.

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2. Some classical mechanics

In this section, we show that the aforementioned geodesic is indeed hyperbolic and has four umbilic points. Let H' denote the left-invariant Hamiltonian on $T^*SO(3)$ associated with a rigid body with distinct moments of inertia $\alpha_3^{-1} > \alpha_2^{-1} > \alpha_1^{-1} > 0$. If we reduce this system with respect to the component of spatial angular momentum corresponding to rotation about a fixed reference axis, we obtain an induced Hamiltonian system on S^2 [20]. The reduced Hamiltonian H and the reduced integral in involution P are given by the formulas [20],[21]:

(1) $H = \alpha_3(x_1\xi_2 - \xi_1 x_2)^2 + \alpha_2(x_1\xi_3 - x_3\xi_1)^2 + \alpha_1(x_3\xi_2 - x_2\xi_3)^2,$

(2) $P = (x_1\xi_2 - \xi_1 x_2)^2 + (x_2\xi_3 - \xi_2 x_3)^2 + (x_3\xi_1 - \xi_3 x_1)^2.$

Here, we identify T^*S^2 with the set of points $\{(x,\xi) \in R^6; |x| = 1, x\cdot\xi = 0\}$. As we show in [20], $H = \sigma(\mathcal{H})$, where $\mathcal{H}$ is a second-order, elliptic differential operator (essentially, the radial part of a left-invariant Laplacian on $SO(3)$), and $P = \sigma(-\Delta_0)$, where $-\Delta_0$ is the standard Laplacian on S^2. Both operators are self-adjoint with respect to the constant curvature metric on S^2.

Proposition 1. *The two geodesics $\Gamma^\pm = \{(x,\xi) \in S^*(S^2); x_2 = \xi_2 = 0\}$ are hyperbolic. Moreover, if $\pi : T^*(S^2) \to S^2$ denotes the standard projection map, then the four points:*

$$\Big(\pm\frac{(\alpha_1-\alpha_2)^{1/2}}{(\alpha_1-\alpha_3)^{1/2}}, 0, \pm\frac{(\alpha_2-\alpha_3)^{1/2}}{(\alpha_1-\alpha_3)^{1/2}}\Big) \in \pi(\Gamma)$$

standard projection map, then the four points:

$$\Big(\pm\frac{(\alpha_1-\alpha_2)^{1/2}}{(\alpha_1-\alpha_3)^{1/2}}, 0, \pm\frac{(\alpha_2-\alpha_3)^{1/2}}{(\alpha_1-\alpha_3)^{1/2}}\Big) \in \pi(\Gamma)$$

are umbilics for the Riemann metric induced on S^2 by H.

Proof. Since H is invariant under reflection in the coordinate planes, it follows that Γ is a geodesic. To show that Γ is hyperbolic, we use symplectic reduction. Reducing the above system at unit momentum with respect to the symplectic S^1 action given by standard geodesic flow on T^*S^2(i.e., the flow of the Hamilton vector field X_P), we get the following reduced system on S^2:

$$h(\Omega) = \alpha_3\Omega_1^2 + \alpha_2\Omega_2^2 + \alpha_1\Omega_3^2, \tag{3}$$

$$p(\Omega) = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = 1. \tag{4}$$

Let $r : T^*S^2 \to S^2$ denote the reduction map. Then, we have $r(\Gamma^{\pm}) = p = (0, \pm 1, 0)$ where $dh(p) = 0$ and d^2h has a saddle point at p. Let Σ be the initial Poincaré cross-section to Γ at p_0, ϕ_t be the flow for X_H and U be a small open neighbourhood of Γ. Then, r maps $U \cap \phi_t(\Sigma)$ diffeomorphically onto a neighbourhood of p and $H \mid_{\phi_t(\Sigma)}$ has a saddle-point at $\phi_t(p_0)$ for any t. Therefore, $\Gamma^{\pm}$ must be hyperbolic. Henceforth, without loss of generality, we put $\Gamma = \Gamma^+$.

To prove that there are four umbilic points lying on $\pi(\Gamma)$, it is best to compare the Riemann metric g induced on T^*S^2 by H, with the standard metric $\tilde{g}$ (induced from R^3) on the triaxial ellipsoid $E^2 = \{(x_1, x_2, x_3) \in R^3; x_1^2/\alpha_1 + x_2^2/\alpha_2 + x_3^2/\alpha_3 = 1\}$, pulled back to S^2 via the homothety $(x_1, x_2, x_3) \to ((\alpha_1)^{1/2}x_1, (\alpha_2)^{1/2}x_2, (\alpha_3)^{1/2}x_3)$. A simple calculation in elliptic-spherical coordinates (see below, [20]) gives:

$$\tilde{g} = \alpha_1\alpha_2\alpha_3\left(\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2}\right)\cdot g.$$

By elementary surface theory, we know that lines of curvature are invariant under non-negative conformal scaling of the metric, and so in particular, the umbilic points of g and $\tilde{g}$ coincide. In the latter case, these points are well-known [1]. q.e.d.

We shall now show that there is a *focusing* effect at these umbilic points on Γ; that is, *all* geodesics on the separatrix Λ intersect Γ at these points. Moreover, the geodesics $\pi(\gamma(t))$ are forwards and backwards asymptotic to Γ. As we shall see later, in the quantum setting there is a corresponding accumulation of L^{∞} norm for the eigenfunctions ψ_n.

We now restrict our attention to the flow, ϕ_t, on the separatrix:

$$\Lambda := \{(x,\xi) \in T^*S^2; H(x,\xi) - \alpha_2 \cdot P(x,\xi) = 0\}.$$

Proposition 2. *Let $\pi(\gamma(t))$ be the base projection of a solution curve $\gamma(t)$ of X_H on Λ. Then, $\pi(\gamma(t))$ intersects Γ at an umbilic point $p_0 = \pi(\gamma(0))$, and there exists T with $\pi(\gamma(T)) = -p_0$, where $-p_0$ is the diametrically opposite umbilic on (S^2, g). Moreover,*

$$\inf_{p\in\Gamma} \|\pi(\gamma(t)) - p\| = \mathcal{O}(e^{-C|t|}),$$

as $|t| \to \infty$. Here, $C > 0$ is a constant, and $\inf \|\cdot\|$ *denotes distance in the metric g.*

Proof. Introduce elliptic-spherical coordinates (u_1, u_2) on S^2, defined by

$$\begin{aligned}
x_1 =& k\, sn(\beta(u_1); k) sn(u_2; k), \\
x_2 =& i\frac{k}{k'} cn(\beta(u_1); k) cn(u_2; k), \\
x_3 =& \frac{1}{k'} dn(\beta(u_1); k) dn(u_2; k),
\end{aligned}$$

where $cn(x;k), sn(x;k), dn(x;k)$ are the basic Jacobian elliptic functions [22], $0 < k < 1$ is the elliptic modulus, and k' is the complementary modulus given by the equation $k^2 + k'^2 = 1$. Moreover, $-\mathbf{K}' \leq u_1 \leq \mathbf{K}', 0 \leq u_2 \leq 4\mathbf{K}$, $\beta(u_1) := \mathbf{K} + i(\mathbf{K}' - u_1)$, and $(x_1, x_2, x_3) \in S^2$, the modular vectors $\mathbf{K}$ and $\mathbf{K}'$ being defined by the elliptic integrals $\int_0^1 \{(1-t^2)(1-k^2t^2)\}^{-1/2} dt$ and $\int_0^1 \{(1-t^2)(1-k'^2t^2)\}^{-1/2} dt$ respectively. Applying Hamilton- Jacobi theory, one finds that the defining equations for $\pi(\gamma(t))$ are:

$$\frac{d\beta(u_1)}{dt} = \frac{[(\alpha_1 - \alpha_2)^{\frac{1}{2}} (sn^2(u_2;k) - \alpha_1)] \cdot (1 - sn^2(\beta(u_1);k))^{\frac{1}{2}}}{sn^2(u_2;k) - sn^2(\beta(u_1);k)}, \tag{5}$$

$$\frac{du_2}{dt} = \frac{[(\alpha_1 - \alpha_2)^{\frac{1}{2}} (sn^2(\beta(u_1);k) - \alpha_1)] \cdot (1 - sn^2(u_2;k))^{\frac{1}{2}}}{sn^2(\beta(u_1);k) - sn^2(u_2;k)}. \tag{6}$$

From equations (5) and (6), we get the integrated Clairault relation:

$$\int_{\beta(u_1(0))}^{\beta(u_1(t))} \omega(x)dx + \int_{u_2(0)}^{u_2(t)} \omega(x)dx = 0, \tag{7}$$

where, $\omega(x) = [(\alpha_1 - \alpha_2)^{\frac{1}{2}} (sn^2(x;k) - \alpha_1)] \cdot (1 - sn^2(x;k))^{-\frac{1}{2}}$. On $\pi(\Gamma)$ we have either $\beta(u_1) \in \{\mathbf{K}, \mathbf{K} + 2i\mathbf{K}'\}$ and u_2 variable, or $u_2 \in \{\mathbf{K}, 3\mathbf{K}\}$ and $\beta(u_1)$ variable. The umbilic points are given by $(\beta(u_1), u_2) \in \{(K, \mathbf{K}), (\mathbf{K}, 3\mathbf{K}), (\mathbf{K}+2i\mathbf{K}', \mathbf{K}), (\mathbf{K}+2i\mathbf{K}', 3\mathbf{K})\}$. We can, without loss of generality, assume that $u_2 = \mathbf{K}$ (the other cases are handled in the same way). So, if $u_2(t) \to \mathbf{K}$ as $t \to T$, it follows from equation (7) that:

$$\int_{\beta(u_1(0))}^{\beta(u_1(t))} \omega(x)dx = C \log |u_2(t) - \mathbf{K}| + \mathcal{O}(1).$$

Since $(\omega \circ \beta)(u_1)$ is in L^1_{loc} away from $\beta(u_1) = \mathbf{K}, \mathbf{K} + 2i\mathbf{K}'$, this forces $\beta(u_1(t)) \to \mathbf{K}$, or $\mathbf{K} + 2i\mathbf{K}'$ as $t \to T$. Since $\beta(u_1)$ and $\dot{u}_2$ have opposite sign, it follows that $\pi(\gamma(t))$ passes through diametrically opposite umbilics. To prove the last assertion, we write down another integrated conservation law:

$$\int_{\beta(u_1(0))}^{\beta(u_1(t))} \eta(x)dx + \int_{u_2(0)}^{u_2(t)} \eta(x)dx = \{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\}^{\frac{1}{2}} \cdot t,$$

where, $\eta(x) = (1 - sn^2(x;k))^{-\frac{1}{2}} = |nc(x;k)|$. This identity, together with the well-known [3] formula, $\int_\alpha^\beta nc(x;k)dx = k'^{-1}[\log(dn(x;k) + k'sn(x;k)) - \log(cn(x;k))]_\alpha^\beta$ gives,

$$\frac{[dn(u_2(t);k) + k'sn(u_2(t);k)][dn(\beta(u_1(t));k) + k'sn(\beta(u_1(t));k)]}{cn(u_2(t);k) \cdot cn(\beta(u_1(t));k)} = C' \exp(Ct),$$

where, $C = k'\{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\}^{\frac{1}{2}}$, and C' is an integration constant. If $t > 0$ and $\pi(\gamma(t))$ is a point not on Γ, it follows that:

$$|cn(u_2(t);k) \cdot cn(\beta(u_1(t));k)| = \mathcal{O}(e^{-Ct}),$$

and thus,

$$\min\{|cn(u_2(t);k)|, |cn(\beta(u_1(t));k)|\} = \mathcal{O}(e^{-\frac{Ct}{2}}). \tag{8}$$

So, for any $t > 0$, either $u_2(t) = \mathbf{K}$, or $3\mathbf{K} + \mathcal{O}(e^{-Ct})$ or $\beta(u_1(t)) = \mathbf{K}$, or $\mathbf{K} + 2i\mathbf{K}' + \mathcal{O}(e^{-Ct})$. Given our characterization of Γ in terms of the coordinates u_2 and u_1, we are done. In the case $t < 0$, replace $cn(x;k)$ by $dn(x;k) + k'sn(x;k)$ in (8), and argue in precisely the same way. q.e.d.

One can prove analogous statements for geodesic flow on the ellipsoid and elliptic billiards in the same way. For the latter two systems, this sort of behaviour is, at least qualitatively, well-known [1].

3. L^∞ estimates for generic eigenfunctions

In this section, we use classical WKB theory, and the following modular identity for the Jacobian sine function $sn(z;k)$ [22],[20]:

$$sn^2(\mathbf{K}+i(\mathbf{K}'-z);k)=k^{-2}\cdot(1-k'^2sn^2(z;k')) \tag{9}$$

to study eigenfunctions. Using the modular relation (9), one can show [20],[8] that the joint eigenfunctions of the operators $\mathcal{H}$ and $-\Delta_0$ are given by the collection of harmonics $\psi_1(u_1)\cdot\psi_2(u_2)$ where ψ_1,ψ_2 satisfy the following Floquet boundary value problems on the real line:

$$\{-\hbar^2\frac{d^2}{dx^2}+k^2sn^2(x;k)\}\psi_2=\lambda(\hbar)\psi_2, \tag{10}$$

$$\psi_2(x+4\mathbf{K})=\psi_2(x),$$

$$\{-\hbar^2\frac{d^2}{dx^2}+k'^2sn^2(x;k')\}\psi_1=(1-\lambda(\hbar))\psi_1, \tag{11}$$

$$\psi_1(x+4\mathbf{K}')=\psi_1(x).$$

Here, (u_1,u_2) denote Jacobian uniformizing variables on S^2 (see [20] and Section 2), $\hbar=\{n(n+1)\}^{-1/2};n=1,2,...$ and for $\hbar$ sufficiently small, we restrict $\lambda(\hbar)$ to the range $0\leq\lambda(\hbar)\leq 1$. By *generic eigenfunctions*, we mean those that are associated with arbitrary energy levels, E (i.e., $1-\lambda(\hbar)\to E$), with $\max\{k'^2sn^2(x;k')\}=k'^2>E>0=\min\{k'^2sn^2(x;k')\}$.

Suppose $1-\lambda(\hbar)\to E$ in this range. Then, it is well known that a given $\psi_1(u_1;\hbar)$ with $\|\psi_1\|_{L^2}=1$ will have L^∞ norm concentrated at the caustics (i.e., the turning points). More precisely [9], if x_0 is a turning point, then, to the right of x_0:

$$\psi_1(u_1;\hbar)=\hbar^{-1/6}[\phi'(u_1)]^{-1/2}Ai[-\hbar^{-2/3}\phi(u_1)]+\mathcal{O}(1) \tag{12}$$

where, $2/3(\phi(x))^{3/2}=\int_{x_0}^{x}[k'^2sn^2(t)-E]^{1/2}dt$, with similar formulas for $x<x_0$. Using well-known [9] asymptotic expansions for the Airy function $Ai(x)$, it follows that,

$$\|\psi(u_1;\hbar)\|_{L^\infty}=C(E)\hbar^{-1/6}+\mathcal{O}(1) \tag{13}$$

Notice that since $1-\lambda(\hbar)<k'^2$, then $\lambda(\hbar)>k^2$. So, in the complementary variable u_2, we pass *over* the potential barrier $k^2sn^2(u_2;k)$, and thus the function $\psi_2(u_2;\hbar)$ has an asymptotic expansion:

$$\psi_2(u_2;\hbar)=e^{i\kappa(u_2)/\hbar}a(u_2;\hbar)+\mathcal{O}(e^{-C/\hbar}). \tag{14}$$

Here, $\kappa(x) = \int_0^x k\, sn(t)dt$, and $a(x;\hbar) \sim a_0(x) + a_1(x)\hbar + ...$ is a classical analytic symbol [18] of order zero. Exactly the same argument works in the case $k^2 < E < 1$ with the roles of u_1 and u_2 reversed. For $1 - \lambda(\hbar) \in [0, C\hbar]$ (i.e., ground state), the expansion (12) for $\psi_1(u_1;\hbar)$ is replaced by:

$$\psi_1(u_1;\hbar) = \hbar^{-1/4} e^{-\kappa(u_1)/\hbar} a(u_1;\hbar) + \mathcal{O}(e^{-C/\hbar}). \tag{15}$$

Again, one can reverse the roles of u_1 and u_2 to get asymptotic formulas corresponding to eigenvalues $1 - \lambda(\hbar) \in [1 - C\hbar, 1 + C\hbar]$. Summing up, we have proved:

Proposition 3. *The L^2-normalized Lamé harmonics $\psi(u_1, u_2;\hbar) := \psi_1(u_1;\hbar) \cdot \psi_2(u_2;\hbar)$ with eigenvalues $\lambda(\hbar) \to E$ where $E \in [0, k^2) \cup (k^2, 1]$ satisfy:*

$$\|\psi(u_1, u_2;\hbar)\|_{L^\infty} = C(E)\hbar^{-1/6} + \mathcal{O}(1),$$

whereas, for $\lambda(\hbar) \in [0, C\hbar]$ and $\lambda(\hbar) \in [1 - C\hbar, 1 + C\hbar]$:

$$\|\psi(u_1, u_2;\hbar)\|_{L^\infty} = C\hbar^{-1/4} + \mathcal{O}(\hbar^{3/4}).$$

4. Microlocal analysis near the potential maximum

We now address the more interesting question of estimating eigenfunctions pointwise near the top of the potential $k^2 sn^2(x;k)$. As we shall see, these eigenfunctions have an accumulation of L^∞ norm in an $\mathcal{O}(\hbar)$ neighbourhood of Γ, with additional accumulation near the umbilic points.

To begin, put $E = k^2$ in equation (10). Since this is a singular energy level for the potential $k^2 sn^2(x;k)$, the standard ansatz of geometric asymptotics [11] breaks down, and we must use other methods. The fundamental construction is the following (see [6] for the C^∞ analogue):

Theorem 1. (Helffer-Sjöstrand [14]) *Let $P(x, \hbar D_x;\hbar)$ be a formal classical analytic pseudodifferential operator of order 0, formally self-adjoint, with symbol defined in a neighbourhood of $(x,\xi) = (0,0)$. Let p be the principal symbol, and assume that p has a non-degenerate saddle point at $(0,0)$ with critical value 0. Then there is a real-valued analytic symbol; $F(t,\hbar) \sim \sum_0^\infty f_j(t)\hbar^j$, defined for t in a neighbourhood of 0, and a formal unitary analytic Fourier integral operator U, whose associated canonical transformation (in the classical sense) is* *defined in a neighbourhood of* $(0,0)$, *and maps this point onto itself, such that microlocally,*

$$U^* F(P,\hbar)U = \frac{1}{2}(x\hbar D_x + \hbar D_x x).$$

The first step in the proof of Theorem 1 is to construct a real-analytic canonical transformation on a sufficiently small open set $\Omega \in (0,0)$ with $\kappa : \Omega \to \Omega$ and $\kappa(0,0) = (0,0)$. This is done in two stages. First, one constructs

$$\kappa_1 : \Omega \to \Omega,$$
$$\kappa_1(0,0) = (0,0),$$

such that,

$$p \cdot \kappa_1(x,\xi) = a(x,\xi)x\xi.$$

By applying a suitable function f to p, we may assume that $a(0,0) = 1$. In the case of a Schrödinger operator $P(\hbar) = -\hbar^2\partial_x^2 + V(x)$ with $V(x)$ real-analytic, $V'(0) = 0$ and $V''(0) < 0$, it is easy to show that:

$$d\kappa_1(0,0) = \kappa_{\pi/4},$$

where κ_θ denotes the rotation by θ in (x,ξ) space. The second step is more difficult, and consists of constructing:

$$\kappa_2 : \Omega \to \Omega,$$
$$\kappa_2(0,0) = (0,0),$$

with the property that,

$$p \cdot \kappa_1 \cdot \kappa_2(x,\xi) = x\xi,$$

with

$$d\kappa_2(0,0) = id.$$

One then associates with $\kappa := \kappa_1 \cdot \kappa_2$ a (formal) unitary analytic Fourier integral operator U_κ of order zero. To finish the proof, one constructs a (formal) analytic *pseudodifferential* operator R of order zero, solving the equation:

$$R \cdot P = \frac{1}{2}(x\hbar D_x + \hbar D_x x) \cdot R.$$

So $U = R \cdot U_\kappa$ and since $d\kappa(0,0) = \kappa_{\pi/4}$, it follows that for sufficiently small x and y, the generating function $\phi(x,y)$ satisfies:

$$\phi(x,y) = \phi_0(x,y) + \mathcal{O}((x,y)^3),$$

where,

$$\phi_0(x,y) = \frac{1}{2}x^2 + \sqrt{2}xy - \frac{1}{2}y^2.$$

Let $\chi \in C_0^\infty$ with support in a sufficiently small neighbourhood Y of $y = 0$ and identically 1 near 0. If $\psi \in \mathcal{D}'(Y)$ then,

$$(16) \qquad U\psi(x;\hbar) = C\hbar^{-1/2} \int e^{i\phi(x,y)/\hbar}\sigma(x,y;\hbar)\chi(y)\psi(y)dy.$$

Here, $x \in X$, a sufficiently small neighbourhood of $x = 0$, and $\sigma(x,y;\hbar)$ is an elliptic, classical analytic symbol of order zero. Since all distributional solutions of the eigenvalue equation $(1/2)(x\hbar D_x + \hbar D_x x)u = \eta' \cdot u$ are linear combinations of $u_+^0(x) = H(x)|x|^{-1/2+i\eta'/\hbar}$ and $u_-^0(x) = H(-x)|x|^{-1/2+i\eta'/\hbar}$, the natural approximate eigenfunctions (defined for $x \in X$) of $P(\hbar)$ solving the equation $(P - \eta)u = 0$ are just $u_+(x;\hbar) := Uu_+^0(x)$ and $u_-(x;\hbar) := Uu_-^0(x)$. These functions will be our basic building blocks. Without loss of generality, we will henceforth work with $u_+(x;\hbar)$. To study this function near $x = 0$, we must estimate the integral:

$$(17)\; u_+(x;\hbar) = \hbar^{-1/2} \int_0^\varepsilon e^{i[\phi(x,y)+\eta' \log y]/\hbar}\sigma(x,y;\hbar)\chi(y) \cdot y^{-1/2}dy,$$

where, $\eta' = F(\eta;\hbar)$, and $supp(\chi) \subset \{y; |y| \le \varepsilon\}$.

Lemma 1. *There exists $\eta \in [-C'\hbar, C'\hbar]$, such that for $|x| \le C\hbar$, $|u_+(x;\hbar)|$ has the uniform asymptotic expansion:*

$$|u_+(x;\hbar)| = C \cdot \hbar^{-1/4} + \mathcal{O}\left(\frac{\hbar^{-1/4}}{\log 1/\hbar}\right),$$

where C is a suitable constant.

Proof. To eliminate the x variable, write:

$$\phi(x,y) = \phi(0,y) + \Phi(x,y),$$

where

$$\Phi(x,y) = \mathcal{O}(x^2) + \mathcal{O}(xy).$$

It follows that:

$$e^{i\Phi(x,y)/\hbar}\sigma(x,y;\hbar) = \sigma(0,y;\hbar) + \mathcal{O}(y) \cdot \sigma(0,y;\hbar) + \mathcal{O}(\hbar).$$

We can thus write the integral in (17) as:

$$C \cdot \hbar^{-1/2} \int_0^\varepsilon e^{i[\phi(0,y)+\eta' \log y]/\hbar}\sigma(0,y;\hbar)\chi(y)y^{-1/2}dy$$
$$+ C\hbar^{-1/2} \int_0^\varepsilon e^{i[\phi(0,y)+\eta' \log y]/\hbar}\chi(y)\sigma'(0,y;\hbar)y^{-1/2}dy + \mathcal{O}(\hbar^{1/2}),$$

where $\sigma'(0,y;\hbar) = \mathcal{O}(y)\cdot\sigma(0,y;\hbar)$. Using the fact that $\phi(0,y) = -y^2+\mathcal{O}(y^3)$, we rescale the second term by introducing the variable $w = \hbar^{-1/2}y$, and do one integration by parts to get:

$$u_+(x;\hbar) = C\hbar^{-1/2}\int_0^{\varepsilon} e^{i[\phi(0,y)+\eta'\log(y)]/\hbar}\chi(y)\sigma(0,y;\hbar)y^{-1/2}dy+\mathcal{O}(\hbar^{1/4})$$
$$= C\hbar^{-1/2}\int_0^{\varepsilon} e^{i[\phi(0,y)+\eta'\log(y)]/\hbar}y^{-1/2}dy+\mathcal{O}(\hbar^{1/4}),$$

since, by the unitarity of U it follows that [15] $\sigma(0,0)=1$. Making a change of variables $w^2 = -\phi(0,y)$, and rescaling the above integral by $z = \hbar^{-1/2}w$, yields

(18)
$$u_+(x;\hbar) = C\hbar^{-1/4}e^{i\frac{\eta'}{2\hbar}\cdot\log\hbar}\int_0^{\varepsilon'\cdot\hbar^{-1/2}} e^{i(-z^2+\frac{\eta'}{\hbar}\log z)}z^{-1/2}dz+\mathcal{O}(\hbar^{1/4}).$$

We now recall a result of März [15] on the structure of the Floquet spectrum in an energy band of size $\mathcal{O}(\hbar)$ about the potential maximum, and then compute the last integral, using a well-known asymptotic expansion for the indefinite Gamma function, $\Gamma(z;\alpha) := \int_0^z e^{-t}t^{\alpha-1}dt$. In [15], März shows that if $|\eta| \le C\hbar$ and $\eta' := F(\eta;\hbar) = \eta+\mathcal{O}(\hbar^2)$ is contained in a gap, then the length of this gap is,

$$\frac{2\hbar}{\log(1/\hbar)}(\arccos[(1+e^{-2\pi\eta'/\hbar})^{-1/2}]) + \mathcal{O}\left(\frac{\hbar}{(\log(1/\hbar))^2}\right).$$

If η' is contained in a band, one replaces arccos by arcsin. So, we choose an eigenvalue η, with:

$$\eta'(\hbar) = \mathcal{O}\left(\frac{\hbar}{\log(1/\hbar)}\right). \tag{19}$$

Putting $r = z^2$, we must evaluate the integral,

$$\int_0^{\varepsilon''\cdot\hbar^{-1}} e^{-ir}\cdot r^{-\frac{3}{4}+\frac{\eta'}{2\hbar}i}dr. \tag{20}$$

Using the asymptotic expansion [3]:

$$\Gamma(z;\alpha) \sim \Gamma(\alpha) - e^{-z}z^{\alpha-1}\left[1+\frac{\alpha-1}{z}+\ldots\right]$$

valid as $z\to\infty$ in $|\arg z| < 3\pi/2$, the lemma follows. q.e.d.

Let us now suppose that we are in the Floquet case, and so, in particular, the potential, $V(x)$, satisfies $V(x+2\pi) = V(x)$. Then, for $\epsilon > 0$ sufficiently small, one can extend (see [15]) u_+ and u_- to functions defined on $(-\epsilon, 2\pi + \epsilon)$, by applying analytic stationary phase outside an arbitrarily small (but fixed) neighbourhood of $x = 0$, and get $u_\pm(x) = \sum_\pm e^{i\phi_\pm(x)/\hbar} a_\pm(x;\hbar) + \mathcal{O}(e^{-c/\hbar})$. Here, $\pm$ denote various microlocal contributions from $\{(x,\xi); \xi^2 + V(x) = E, \xi > 0\}$, and $\{(x,\xi); \xi^2 + V(x) = E, \xi < 0\}$ respectively. Furthermore, $u_\pm$ can be constructed so as to satisfy,

$$(P - \eta)u_\pm(x;\hbar) = \mathcal{O}(e^{-c/\hbar})$$

pointwise, on such an interval. In [15], März derives a formula for the approximate translation matrix, $\tilde{T}(\eta;\hbar)$ corresponding to the basis, $u_\pm(x;\hbar)$, which, as it turns out, is within $\mathcal{O}(e^{-c/\hbar})$ of the exact translation matrix, $T(\eta;\hbar)$. Applying the Floquet condition,

$$\operatorname{Trace} \tilde{T}(\eta;\hbar) = \pm 2$$

one readily verifies that, for $|\eta| \leq C\hbar$, the eigenfunctions (up to rescaling) of $\tilde{T}(\eta;\hbar)$ must be of the form:

$$u_+(x;\hbar) \pm u_-(x;\hbar) + \mathcal{O}(e^{-c/\hbar}).$$

If we require the symmetry condition, $V(x) = V(-x)$, then, it is well-known that Floquet eigenfunctions fall into four categories: that is, each eigenfunction solves one of four distinct Sturm-Liouville boundary-value problems. By Lemma 1, the functions of interest to us are $u_+(x;\hbar) + u_-(x;\hbar)$. To show that there are true eigenfunctions close to these functions, we argue as follows. Since, $V(-x) = V(x)$, it is not difficult to show that the canonical transformation $\kappa : \Omega \to \Omega$ is odd, and thus, $\phi(-x,-y) = \phi(x,y)$. It follows that,

$$u_+(0;\hbar) = u_-(0;\hbar). \tag{21}$$

Furthermore, if $u(x;\hbar)$ is a solution of the Schrödinger equation on $(-\pi - \epsilon, 3\pi + \epsilon)$, we have the well-known pointwise estimate (see, for example, [15, Lemma 7.1]):

$$(|\hbar\partial_x u(x)|^2 + |u(x)|^2)^{1/2} \leq C_\epsilon e^{\epsilon/\hbar}(|\hbar\partial_y u(y)|^2 + |u(y)|^2)^{1/2}. \tag{22}$$

Here, $y \in [-\pi, 3\pi]$, and $x \in [y, 3\pi]$, and $\epsilon > 0$ is arbitrary. Combining (21), (22), together with the characterization of the Floquet eigenfunctions as solutions of Sturm-Liouville boundary-value problems, yields that there exist eigenfunctions $u(x;\hbar)$ corresponding to eigenvalues $\eta = \mathcal{O}(\hbar/|\log\hbar|)$, with,

$$\|u - (u_+ + u_-)\|_{L^\infty} = \mathcal{O}(e^{-c/\hbar}). \tag{23}$$

We shall now address the question of L^2-normalization (see also [6, Proposition 14]). Since, $u_\pm$ have standard WKB expansions outside a fixed neighbourhood, $\{x; |x| \leq \epsilon\}$, it follows that,

$$\int_{\pi-\epsilon\geq|x|\geq\epsilon} |u_+(x;\hbar) + u_-(x;\hbar)|^2 dx = \mathcal{O}(1). \tag{24}$$

To compute the L^2-norm inside $\{x; |x| \leq \epsilon\}$, it is useful to note that U is a microlocally, unitary Fourier integral operator on a sufficiently small open neighbourhood, Ω, containing $(0,0)$. Furthermore, the microsupport of $u_\pm(x;\hbar)$ is contained in Ω for x sufficiently small. It therefore follows, modulo terms that are $\mathcal{O}(e^{-c/\hbar})$, that:

$$\int_{|x|\leq\epsilon} |u_\pm(x;\hbar)|^2 dx = C\hbar^{-1}\int_0^\epsilon d\xi \,|\int_0^\infty e^{-\frac{i}{\hbar}(x\xi-\eta\log x)}\chi(x)x^{-\frac{1}{2}}dx|^2$$

Estimating this last integral, leads to that, for $|\eta| \leq C\hbar/|\log\hbar|$,

$$\int_{|x|\leq\epsilon} |u_\pm(x;\hbar)|^2 dx = C\log\left(\frac{1}{\hbar}\right) + \mathcal{O}(1)$$

and,

$$\int_{|x|\leq\epsilon} u_+(x;\hbar)\overline{u_-(x;\hbar)}dx = \mathcal{O}(1).$$

Summing up, we have proved:

Proposition 4. *Suppose, $V(-x) = V(x)$, and η is a Floquet eigenvalue with $|\eta| \leq C\hbar\big/|\log\hbar|$. Then, there exists a subsequence of η's, such that, for $|x| \leq C\hbar$, the corresponding L^2-normalized eigenfunctions satisfy,*

$$|u(x;\hbar)| = C\frac{\hbar^{-1/4}}{|\log\hbar|^{1/2}} + \mathcal{O}\left(\frac{\hbar^{-1/4}}{|\log\hbar|^{3/2}}\right).$$

5. Eigenfunction localization along Γ

We are now ready to prove our main result:

Theorem 2. *Let $\Gamma(n^{-1}); n = 1,2,3,..$ denote a tube of width $\mathcal{O}(n^{-1})$ about Γ, and let $V_j; j = 1,2,3,4$ denote arbitrarily small (but fixed) disconnected neighbourhoods about the four umbilic points. Then, there exists a sequence of Lamé harmonics $\psi(u_1,u_2;n) := \psi_1(u_1;n)\cdot\psi_2(u_2;n); n = 1,2,...,$ such that:*

$$\|\psi\|_{L^\infty(\Gamma(n^{-1})-\bigcup_j V_j)} = C\frac{n^{1/4}}{\log n} + \mathcal{O}\left(\frac{n^{1/4}}{(\log n)^2}\right), \tag{25}$$

$$\|\psi\|_{L^\infty(n^{-1}V_j)} = C\frac{n^{1/2}}{\log n} + \mathcal{O}\left(\frac{n^{1/2}}{(\log n)^2}\right), \tag{26}$$

noindent and $\|\psi\|_{L^\infty} = \mathcal{O}(1/\log n)$ *outside an arbitrarily small (but fixed) neighbourhood of* Γ. *Here,* $C > 0$ *is a constant.*

Proof. Let $\hbar = [n(n+1)]^{-1/2}; n = 1,2,\ldots$ and $\psi_1(u_1;\hbar)$ be an eigenfunction of (11) with eigenvalue

$$1 - \lambda(\hbar) \in [k'^2 - C\hbar/|\log\hbar|, k'^2 + C\hbar/|\log\hbar|].$$

The corresponding eigenvalue $\lambda(\hbar)$ of $\psi_2(u_2;\hbar)$ also lies in the interval $[k^2 - C\hbar/|\log\hbar|, k^2 + C\hbar/|\log\hbar|]$. So we are working at the top of the potential in *both* Floquet problems. If we denote $X := \{\Gamma(n^{-1}) - \bigcup_{j=1}^4 V_j\}^c$, then

(27)
$$X \subseteq \{(u_1,u_2) \in S^2;\ -\mathbf{K}' + \epsilon \le u_1 \le \mathbf{K}' - \epsilon,$$
$$\mathbf{K} - \epsilon\hbar \le u_2 \le K + \epsilon\hbar\}.$$

Here, $\epsilon > 0$ is sufficiently small, and ^{c} denotes the connected component with $z > 0$ (the other cases are handled in the same fashion). Since $k^2 sn^2(u_2;k)$ has a non-degenerate maximum at $u_2 = \mathbf{K}$, from Proposition 4 it follows that for $(u_1,u_2) \in X$,

$$|\psi_2(u_2;n)| = C\frac{n^{1/4}}{(\log n)^{1/2}} + \mathcal{O}\left(\frac{n^{1/4}}{(\log n)^{3/2}}\right),$$

and moreover,

$$\|\psi_1(u_1;n)\|_{L^\infty(X)} = C\frac{1}{(\log n)^{1/2}} + \mathcal{O}\left(\frac{1}{n(\log n)^{1/2}}\right).$$

The first part of Theorem 2 then follows, since $\psi(u_1,u_2;n) = \psi_1(u_1;n)\cdot\psi_2(u_2;n)$. To prove the second part of the theorem, note that *both* Floquet potentials $k^2 sn^2(u_1;k)$ and $k'^2 sn^2(u_2;k')$ attain their maxima at the umbilic points $u_1 = \pm\mathbf{K}', u_2 = K, 3\mathbf{K}$, and apply Proposition 4. q.e.d.

Remark. Suppose,

$$\alpha_2 = \frac{1}{2}(\alpha_1 + \alpha_3).$$

In this case, we have

$$k^2 = k'^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3} = \frac{1}{2}.$$

It is well-known [8],[20] that the Lamé Schrödinger operator $P(\hbar) = -\hbar^2\partial_x^2 + \frac{1}{2}sn^2(x)$ has $\frac{1}{2}$ as an eigenvalue (i.e., an eigenvalue precisely at the potential maximum, *independent* of $\hbar$), provided we restrict $\hbar$ to the subsequence $\{[2n(2n+1)]^{-1/2}; n = 0, 1, 2, ...\}$. Shifting the potential down by $\frac{1}{2}$ in order to adhere to our convention, we put:

$$\eta' = F(0; \hbar) \sim f_2(0)\hbar^2 + f_3(0)\hbar^3 + ...$$

since $f_0(0) = f_1(0) = 0$. Thus, by the asymptotic expansion for the indefinite Gamma function (see above), we obtain that:

$$\int_0^{\epsilon''\cdot\hbar^{-1}} e^{-ir} \cdot r^{\frac{-3}{4} + \frac{\eta'}{2\hbar}i} dr = \Gamma(1/4) + \mathcal{O}(\hbar^{3/4}).$$

This implies, by the estimates in Lemma 1, and Proposition 4, that the error in (25) is improved to $\mathcal{O}(n^{-1/4}/\log n)$, whereas the error in (26) is $\mathcal{O}(1/\log n)$.

Remark. In [19], Seeger and Sogge show that for a given self-adjoint, elliptic pseudodifferential operator $P \in \Psi^m_{1,0}(M)$, with strictly convex principal symbol $p(x,\xi)$, there is a universal upper bound for the L^2-normalized eigenfunctions, given by:

$$\|\phi_\lambda\|_{L^\infty} \leq C\lambda^{\frac{n-1}{2m}},$$

where, ϕ_λ is an eigenfunction corresponding to the eigenvalue λ, and $n = \dim M$. In our case, it is plausible that the upper bound is attained by the eigenfunctions associated with Γ at the four umbilic points $(\pm p_0, \pm p_1)$. This would imply that the the actual upper bound is, ever so slightly, better than the universal Seeger-Sogge prediction. However, there are gaps in the asymptotics of $u_\pm$ which must be worked out if one is to prove this rigorously.

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